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Dedekind Sums: Properties and Applications to Number Theory and Lattice Point Enumeration

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I affirm that I have adhered to the Honor Code in this Assignment.

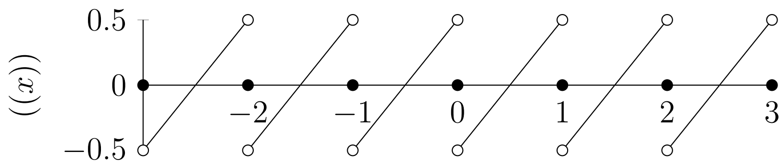
1 Introduction

Dedekind sums are defined [16, p. 1] as follows for integers a, b with $b \geq 1$:

$$s(a,b) = \sum_{k \bmod b} \left(\left(\frac{k}{b}\right)\right)\left(\left(\frac{ak}{b}\right)\right),$$

where for $x \in \mathbb{R}$ the sawtooth function is defined as

$$((x)) = \begin{cases} 0 & x \in \mathbb{Z} \\ x - \lfloor x \rfloor - \frac{1}{2} & \text{otherwise.} \end{cases}$$



In this paper, we will explore the origins, applications, and properties of these sums and one of their generalizations. We seek to explain what these sums represent and how they behave by exploring some of their arithmetic properties. In addition, we hope to show the reader why one should care about these sums. We do this by presenting two important areas in which these sums appear: number theory and the study of enumerating lattice points inside of polytopes. While these are two of the most prominent areas in which Dedekind sums appear, they are by no means the only ones. Dedekind sums appear in areas as diverse as computer science [10], topology [9], geometry [2], and many others.

The first section explores the arithmetic properties of Dedekind sums. We start by showing some basic properties which help us understand the nature of the sums as a function of two integers or one rational number. We then state the reciprocity theorem which sits at the heart of understanding the sums. Using this reciprocity theorem, we show a deep connection to continued fractions and the Euclidean algorithm, before finishing with another application of the reciprocity theorem which adds insight into the behavior of the sums.

In the second section, we motivate Dedekind sums with applications and connections to two different areas of mathematics. First, we explore the connection with various number theory topics, including the theory of quadratic residues and the Dedekind η function which is the origin of Dedekind sums. Secondly, we show how Dedekind sums and a natural generalization, the Fourier-Dedekind sums, are key in counting lattice points inside of polytopes.

We conclude by exploring some recent efforts to answer open questions about specific values of Dedekind sums.

1.1 Fundamental Properies

We start by stating the following elementary and useful proposition.

Proposition 1.

(i) $s(-a,b) = -s(a,b)$

(ii) If $a \equiv a' \pmod b$ then $s(a,b) = s(a',b)$

(iii) If $aa' \equiv 1 \pmod b$, then $s(a,b) = s(a',b)$.

Proof. (from [16, p. 26][1]) (i) and (ii) follow from the oddness and periodicity of the sawtooth function. We can see (iii) through the following simple computation. Since a' is relatively prime to b, multiplying $\{0,1,2,\ldots,b-1\}$ by a' just permutes these so we can replace k with $a'k$ in our sum:

$$\begin{aligned} s(a,b) &= \sum_{k \bmod b} \left(\left(\frac{k}{b}\right)\right)\left(\left(\frac{ak}{b}\right)\right) \\ &= \sum_{k \bmod b} \left(\left(\frac{a'k}{b}\right)\right)\left(\left(\frac{aa'k}{b}\right)\right) \\ &= \sum_{k \bmod b} \left(\left(\frac{k}{b}\right)\right)\left(\left(\frac{a'k}{b}\right)\right) = s(a',b). \end{aligned}$$

□

Note that (ii) shows that $s(a,b)$ is periodic in a with period b. This periodicity is very similar to the periodicity of the greatest common divisor function. We will see that there are many similarities between these two functions.

It turns out that for b prime, the above relations (ii) and (iii) have a converse. Namely, if $s(a,b) = s(a',b)$ then either $a \equiv a' \pmod p$ or $aa' \equiv 1 \pmod p$ [4, p. 162]. We will later discuss this and the interesting ways that the converse fails for composite moduli, b.

Now, we state a useful lemma about the sawtooth function.

Lemma 1. For any $x \in \mathbb{R}$, $\sum_{k=0}^{b-1}\left(\left(\frac{k+x}{b}\right)\right) = ((x))$.

Proof. (from [16]) We consider the function $D(x) = \sum_{k=0}^{b-1}\left(\left(\frac{k+x}{b}\right)\right) - ((x))$. We know $D(x+m) = D(x)$ for integers m since $((x+m)) = ((x))$ and replacing x by $x+m$ in $(((k+x)/b))$ just shifts the index of the sum and leaves the total sum unchanged. So, we restrict $0 \le x < 1$. When $x = 0$,

$$D(0) = \sum_{k=0}^{b-1}\left(\left(\frac{k}{b}\right)\right) = \sum_{k=1}^{b-1}\left(\frac{k}{b} - \frac{1}{2}\right) = \frac{b(b-1)}{2b} - \frac{b-1}{2} = 0.$$

[1] During much of this honors project and resulting paper, the Carus Mathematical Monographs on Dedekind Sums by Hans Rademacher and Emil Grosswald [16] served as a guide. Since it was published in 1972 following the death of Professor Redamacher, the knowledge about Dedekind sums has been greatly expanded. In this paper, we seek to both highlight what we find to be the most interesting parts of this book while extending the theory to include the more modern results and rich avenues of research that have followed.

When $0 < x < 1$, we write $D(0) = \sum_{k \bmod b} \left(\left(\frac{k+x}{b}\right)\right) - ((x))$. Similar to above, we pick representatives k for each of the residue classes mod b such that $0 < \frac{k+x}{b} < 1$. Then we have

$$D(0) = \sum_{k=0}^{b-1} \left(\frac{k+x}{b} - \frac{1}{2} \right) - \left(x - \frac{1}{2} \right) = \frac{b(b-1)}{2b} + \frac{xb}{b} - \frac{b}{2} - x + \frac{1}{2} = \frac{b-1}{2} - \frac{b-1}{2} = 0.$$

□

We can use Lemma 1 to simplify Dedekind sums further with the following slightly surprising result.

Proposition 2. For any integer $m > 0$, $s(am, bm) = s(a, b)$.

This shows that for rational numbers $x = \frac{a}{b}$ we can define the Dedekind sum $s(x) = s(a, b)$ without worrying about how the fraction x is expressed.

Proof. (from [17]) In the following sum, express $k = \alpha b + \beta$ with α, β integers and $0 \le \beta < b$. Then we have

$$\begin{aligned} s(am, bm) &= \sum_{k=0}^{bm-1} \left(\left(\frac{k}{bm} \right)\right) \left(\left(\frac{kam}{bm} \right)\right) \\ &= \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{b-1} \left(\left(\frac{\alpha b + \beta}{bm} \right)\right) \left(\left(\frac{(\alpha b + \beta) a}{b} \right)\right) \\ &= \sum_{\alpha=0}^{m-1} \sum_{\beta=0}^{b-1} \left(\left(\frac{\alpha b + \beta}{bm} \right)\right) \left(\left(\frac{\beta a}{b} \right)\right) \\ &= \sum_{\beta=0}^{b-1} \left(\left(\frac{\beta a}{b} \right)\right) \sum_{\alpha=0}^{m-1} \left(\left(\frac{\alpha + \beta/b}{m} \right)\right). \\ &= \sum_{\beta=0}^{b-1} \left(\left(\frac{\beta a}{b} \right)\right) \left(\left(\frac{\beta}{b} \right)\right) = s(a, b). && \text{(by Lemma 1)} \end{aligned}$$

□

With this basic understanding of the behavior of Dedekind sums, we can state the previously mentioned reciprocity theorem.

Theorem 1. [Dedekind reciprocity] Let a and b be coprime integers. Then

$$s(a, b) + s(b, a) = -\frac{1}{4} + \frac{1}{12} \left(\frac{a}{b} + \frac{b}{a} + \frac{1}{ab} \right).$$

There are many proofs of this including those using elementary methods, those using analytic methods, one peculiar one using Stieltjes integrals, and two that we will see later in this paper (see e.g. [1, p. 62], [4, p. 153], [16]).

One important result of Theorem 1, along with Proposition 1, is that Dedekind sums can be computed using the Euclidean algorithm in the same way that the greatest common divisor is calculated. We know that this runs quickly ($\mathcal{O}(\log(b))$ for calculating $s(a,b)$ [4, p. 7]). In the next section, we will see how this idea can be exploited to find a representation of the Dedekind sum in terms of continued fractions.

Theorem 1 is central to the study and use of Dedekind sums. Although it is still widely used, it has been generalized many times. The most cited one is the following three term relation due to Redemacher [15].

Theorem 2. Let a, b and c be pairwise coprime integers and let a', b' and c' be such that $aa' \equiv 1 \pmod{bc}, bb' \equiv 1 \pmod{ac}, cc' \equiv 1 \pmod{ab}$. Then

$$s(bc', a) + s(ca', b) + s(ab', c) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}\right).$$

We say that this is a generalization since letting $c = 1$ gives Theorem 1. In addition, as recently as 2015, Xiaoying Du And Lei Zhang [5] found another reciprocity formula and proved it using analytic methods and Dirichlet L-functions.

Theorem 3. Let a, b be coprime integers with $aa' \equiv 1 \pmod{b}$ and $bb' \equiv 1 \pmod{a}$.

$$12(s(2b', a) + s(2a', b)) = \frac{a^2 + b^2 + 4}{24ab} - \frac{1}{4}.$$

Theorems 2 and 3 are just two of many similar results which appear regularly in the literature. The ubiquity of these sorts of identities demonstrate the surprising symmetry that Dedekind sums have. These relations are often very helpful when working with Dedekind sums.

We have seen that $s(a,b)$ is periodic in the first term with period b. Another implication of Theorem 1 shows that $s(a,b)$ is roughly linear with periodic (with period a) fluctuations in the second term. Specifically, we have the following proposition which does not seem to appear in the literature.

Proposition 3. For fixed $a > 0$ and $b > a$, write $b = aq + r$ with $0 < r \leq a$. Let $d = (a,b)$. Then we have

$$s(a,b) = s(a,r) + \frac{q}{12} - \frac{a^2 + d^2}{12r(a + \frac{r}{q})}. \tag{1}$$

In particular, for all $b > 1$,

$$\left| s(a,b) - s(a,r) - \frac{q}{12} \right| \leq \frac{a}{6}. \tag{2}$$

Proof. Let $a = a'd, b = b'd, r = r'd$ so $b' = a'q + r'$. This means $(a', b') = (a', r') = 1$.

Applying Theorem 1 and Proposition 2 each twice, we get

$$\begin{aligned}
s(a,b) &= s(a', r' + qa') \\
&= -s(r' + qa', a') - \frac{1}{4} + \frac{1}{12}\left(\frac{a'}{r' + qa'} + \frac{r' + qa'}{a'} + \frac{1}{a'(r' + qa')}\right) \\
&= -s(r', a') - \frac{1}{4} + \frac{1}{12}\left(\frac{a'}{r' + qa'} + \frac{r'}{a'} + q + \frac{1}{a'(r' + qa')}\right) \\
&= s(a', r') + \frac{1}{4} - \frac{1}{12}\left(\frac{a'}{r'} + \frac{r'}{a'} + \frac{1}{a'r'}\right) - \frac{1}{4} + \frac{1}{12}\left(\frac{a'}{r' + qa'} + \frac{r'}{a'} + q + \frac{1}{a'(r' + qa')}\right) \\
&= s(a', r') + \frac{q}{12} + \frac{1}{12}\left(\frac{a'^2 + 1}{a'(r' + qa')} - \frac{a'}{r'} - \frac{1}{a'r'}\right) \\
&= s(a, r) + \frac{q}{12} + \frac{1}{12}\left(\frac{a^2 + d^2}{a(r + qa)} - \frac{a}{r} - \frac{d^2}{ar}\right) \\
&= s(a, r) + \frac{q}{12} - \frac{q(a^2 + d^2)}{12r(aq + r)} \\
&= s(a, r) + \frac{q}{12} - \frac{a^2 + d^2}{12r(a + \frac{r}{q})}.
\end{aligned}$$

To see the bound in (2), observe two cases. If $1 < b \leq a$, then $s(a,b) = s(a,r)$ and $q = 0$ so it holds trivially. When $b > a$, (1) gives

$$\left|s(a,b) - s(a,r) - \frac{q}{12}\right| = \left|\frac{a^2 + d^2}{12r(a + \frac{r}{q})}\right| \leq \frac{2a^2}{12a} = \frac{a}{6}$$

since $d \leq a$, $r \geq 1$ and $a + \frac{r}{q} \geq a$.

$\square$

Note that this bound is at least somewhat tight. For example, when $b = a + 1$ and a is relatively large,

$$\frac{a^2 + d^2}{12r(a + \frac{r}{q})} \approx \frac{a}{12}.$$

We can also see this linearity in b in another way. In particular it is not hard [1, p. 62] to calculate directly that

$$\begin{aligned}
s(1,b) &= \frac{(b-1)(b-2)}{12b} \\
s(2,b) &= \frac{(b-1)(b-5)}{24b}. \qquad \text{(for odd } b\text{)}
\end{aligned}$$

Proving these can be done easily using Theorem 1.

This concludes our discussion of the fundamental properties of Dedekind sums. We have seen that Dedekind sums can be thought of as either a function of a single rational number, or as a periodic function of two integers. In the next section, we will use these ideas to explicitly show the connection between the Euclidean algorithm and evaluating Dedekind sums.

1.2 What Values do Dedekind Sums Take?

In [16, p. 28], it is left as an open question whether the following 2 theorems (which have since been proved) are true.

Theorem 4. The set $\{s(a,b) : a,b \in \mathbb{Z}\}$ is dense in $\mathbb{R}$.

Theorem 5. The set
$$\left\{\left(\frac{a}{b}, s(a,b)\right) \in \mathbb{R}^2 : a,b \in \mathbb{Z}, b \neq 0\right\},$$
the graph of Dedekind sums, is dense in $\mathbb{R}^2$.

In this section, we present the work of Dean Hickerson which proves these two theorems, while shedding more insight onto how Dedekind sums act through a connection to continued fractions. Clearly, Theorem 5 implies Theorem 4 so we will just prove the former.

For a brief introduction to continued fractions, we follow [14, chapter 7]. First, we apply the Euclidean algorithm for u_0, u_1 relatively prime.

$$\begin{aligned} u_0 &= u_1 a_0 + u_2 \\ u_1 &= u_2 a_1 + u_3 \\ &\vdots \\ u_{j-1} &= u_j a_{j-i} + u_{j+1} \\ u_j &= u_{j+1} a_j. \end{aligned} \tag{3}$$

with $u_1 > u_2 > \ldots, u_{j+1}$ and $a_i > 0$ for all $i \geq 1$. Letting $\zeta_i = \frac{u_i}{u_{i+1}}$ and re-writing each equation from the Euclidean algorithm, we see that $\zeta_i = a_i + \frac{1}{\zeta i+1}$. So, putting them together gives
$$\frac{u_0}{u_1} = \zeta_0 = a_1 + \frac{1}{\zeta_1} = a_1 + \frac{1}{a_2 + \frac{1}{\zeta_2}}$$
This can be extended in the natural way. We let $\langle a_0, a_1, \ldots, a_j\rangle$ denote this simple continued fraction.

One can see that for $a_i > 0$ and $i \geq 1$, these simple continued fractions uniquely determine all rational numbers [14]. Now, we show how Hickerson uses the simple continued fraction decomposition of $\frac{h}{k}$ with $(h,k) = 1$ to compute $s(h,k)$.

Theorem 6. [7] Let $\langle a_0, \ldots, a_r\rangle$ be a simple finite continued fraction. Then,

$$s(\langle a_0, \ldots, a_r\rangle) = \frac{-1+(-1)^r}{8} + \frac{1}{12}(\langle 0, a_1, \ldots, a_r\rangle + (-1)^{r+1}\langle 0, a_r, \ldots, a_1\rangle + a_1 - a_2 + \cdots + (-1)^{r+1}a_r) \tag{4}$$

Proof. We proceed by induction on r.

When $r = 0$, the LHS of (4) is $s(\langle a_0\rangle) = s\left(\frac{a_0}{1}\right) = 0$. And, the right hand side of (4) is also 0.

Now, assume that (4) holds for $r-1$ for $r \geq 1$. Since $s(x)$ has period one, we have $s(\langle a_0, \dots, a_r\rangle) = s(a_0 + \langle 0, a_1 \dots, a_r\rangle) = s(\langle 0, \dots, a_r\rangle)$. Now, let $\frac{h}{k} = \langle 0, \dots, a_r\rangle$ with $(h,k) = 1$. Theorem 1 gives

$$s(\langle 0, a_1 \dots, a_r\rangle) = -\frac{1}{4} + \frac{1}{12}\left(\langle 0, \dots, a_r\rangle + \frac{1}{\langle 0, a_1, \dots, a_r\rangle} + \frac{1}{hk}\right) - s\left(\frac{1}{\langle 0, a_1, \dots, a_r\rangle}\right)$$
$$s(\langle 0, a_1 \dots, a_r\rangle) = -\frac{1}{4} + \frac{1}{12}\left(\langle 0, \dots, a_r\rangle + \langle a_1, \dots, a_r\rangle + \frac{1}{hk}\right) - s(\langle a_1, \dots, a_r\rangle)$$

Since $\frac{1}{\langle 0, x_1, \dots, x_n\rangle} = \langle x_1, \dots, x_n\rangle$ by the construction of continued fractions. Now we apply the induction hypothesis to get

$$\begin{aligned}
s(\langle 0, a_1 \dots, a_r\rangle) &= -\frac{1}{4} + \frac{1}{12}\left(\langle 0, \dots, a_r\rangle + \langle a_1, \dots, a_r\rangle + \frac{1}{hk}\right) - \frac{-1+(-1)^{r-1}}{8} \\
&\quad - \frac{1}{12}\left(\langle 0, a_2, \dots, a_r\rangle + (-1)^r\langle 0, a_r, \dots, a_2\rangle + a_2 - a_3 + \dots + (-1)^r a_r\right) \\
&= -\frac{-1+(-1)^{r-1}}{8} + \frac{1}{12}\left(\langle 0, a_1, \dots, a_r\rangle + \frac{1}{hk} + a_1 + \langle 0, a_2, \dots, a_r\rangle\right) \\
&\quad - \frac{1}{12}\left(\langle 0, a_2, \dots, a_r\rangle + (-1)^r\langle 0, a_r, \dots, a_2\rangle + a_2 - a_3 + \dots + (-1)^r a_r\right) \\
&= -\frac{-1+(-1)^{r-1}}{8} + \frac{1}{12}\Big(\langle 0, a_1, \dots, a_r\rangle + \frac{1}{hk} + (-1)^{r+1}\langle 0, a_r, \dots, a_2\rangle \\
&\quad + a_1 - a_2 + \dots (-1)^{r+1} a_r\Big)
\end{aligned}$$

So, now it just remains to show that $\langle 0, a_r, \dots, a_1\rangle = \langle 0, a_r, \dots, a_2\rangle + \frac{(-1)^{r+1}}{hk}$.

Let $b_i = a_{r+1-i}$. Then, we need to show that

$$\langle 0, b_1, \dots, b_r\rangle = \langle 0, b_1, \dots, b_{r-1}\rangle = \frac{(-1)^{r+1}}{hk}. \tag{5}$$

Note that $\frac{h}{k} - \langle 0, a_1, \dots, a_r\rangle = \langle 0, b_r, \dots, b_1\rangle$.

Now, let the convergents (as defined in [14]) to $\langle 0, b_1, \dots, b_r\rangle$ be given by $\frac{h_i}{k_i}$ so $\langle 0, b_1, \dots, b_i\rangle = \frac{h_i}{k_i}$. So, in particular, $\langle 0, b_1, \dots, b_r\rangle = \frac{h_r}{k_r}$ and $\langle 0, b_1, \dots, b_{r-1}\rangle = \frac{h_{r-1}}{k_{r-1}}$.

We claim [14, Section 7.3] that $\frac{k_r}{k_{r-1}} = \langle b_r, \dots, b_1\rangle$ which is the same as $\langle a_1, \dots, a_r\rangle = \frac{k}{h}$. So we have that $k = k_r$ and $h = k_{r-1}$ since both of these fractions are reduced.
So, (5) becomes $\frac{h_r}{h_k} - \frac{h_{r-1}}{k_{r-1}} = \frac{(-1)^{r+1}}{k_r k_{r-1}}$. This result about the difference between successive convergents is well known (e.g. [14, p. 330]) which completes the proof. □

This shows that the Dedekind sum depends heavily on the late terms in the continued fraction decomposition. So, while these late terms don't affect the value $\frac{h}{k}$ much, they do have a significant effect on $s(h,k)$. This is illustrated well in the following proof.

Proof of Theorem 5. (from [7]) Let $(x,y) \in \mathbb{R}^2$ and $\varepsilon > 0$ be chosen arbitrarily. We need to find a rational number $\frac{h}{k} \in \mathbb{Q}$ such that

$$\left|x - \frac{h}{k}\right| + |y - s(h,k)| < \varepsilon. \tag{6}$$

We can assume that x is irrational and y is rational. As shown in [14], every irrational x has an infinite simple continued fraction which we denote $\langle b_0, b_1, \ldots\rangle$.

Now, for any positive integer s and any real $\alpha > 0$, we look at the sth convergent to x. By the triangle inequality and some manipulation we have

$$\begin{aligned}
|x - \langle b_0, b_1, \ldots, b_{s-1}, \alpha\rangle| &\le |x - \langle b_0, b_1, \ldots, b_{s-1}\rangle| + |\langle b_0, b_1, \ldots, b_{s-1}\rangle - \langle b_0, b_1, \ldots, b_{s-1}, \alpha\rangle| \\
&= \left|x - \frac{h_{s-1}}{k_{s-1}}\right| + \left|\frac{h_{s-1}}{k_{s-1}} - \frac{\alpha h_{s-1} + h_{s-2}}{\alpha k_{s-1} + k_{s-2}}\right| \\
&= \left|x - \frac{h_{s-1}}{k_{s-1}}\right| + \left|\frac{h_{s-1}(\alpha k_{s-1} + k_{s-2}) - k_{s-1}(\alpha h_{s-1} + h_{s-2})}{k_{s-1}(\alpha k_{s-1} + k_{s-2})}\right| \\
&= \left|x - \frac{h_{s-1}}{k_{s-1}}\right| + \left|\frac{h_{s-1}k_{s-2} - k_{s-1}h_{s-2}}{k_{s-1}(\alpha k_{s-1} + k_{s-2})}\right| \\
&= \left|x - \frac{h_{s-1}}{k_{s-1}}\right| + \frac{1}{k_{s-1}(\alpha k_{s-1} + k_{s-2})} \\
&\le \left|x - \frac{h_{s-1}}{k_{s-1}}\right| + \frac{1}{k_{s-1}k_{s-2}}.
\end{aligned}$$

As $s \to \infty$, the convergents go to 0 while k_{s-1} and $k_{s-2} \to \infty$. Since there is no α dependence in the last expression above, we can find an s such that for all $\alpha > 0$,

$$|x - \langle b_0, b_1, \ldots, b_{s-1}, \alpha\rangle| < \varepsilon/2. \tag{7}$$

Similarly $x-12y$ is irrational so we may find an infinite simple continued fraction $x-12y = \langle d_0, d_1 \ldots\rangle$ and we may find a large positive integer t such that

$$|x - 12y - \langle d_0, d_1, \ldots, b_{t-1}, \alpha\rangle| < \varepsilon. \tag{8}$$

Clearly, we can find such a pair s, t such that $s + t$ is even. Now find positive integers m and n such that

$$b_1 - b_2 + \cdots + (-1)^s b_{s-1} + (-1)^{s+1} m + (-1)^{s-1} n + (-1)^{t-1} d_{t-1} + \cdots - d_1 = b_0 - d_0.$$

Now, define $\frac{h}{k} = \langle b_0, b_1, \ldots, b_{s-1}, m, n, d_{t-1}, \ldots, d_1\rangle$. The total number of integers in this continued fraction is $1 + t + s + 2$ which is odd for $t + s$ even so we have by Theorem 6,

$$\begin{aligned}
s\left(\frac{h}{k}\right) = \frac{1}{12}\Bigg(&\underbrace{\langle 0, b_1, \ldots, b_{s-1}, m, n, d_{t-1}, \ldots, d_1\rangle}_{(*)} - \underbrace{\langle 0, d_1, \ldots, d_{t-1}, n, m, b_{s-1}, \ldots, b_1\rangle}_{(**)} \\
&+ \underbrace{b_1 - b_2 + \cdots + (-1)^s b_{s-1} + (-1)^{s+1} m + (-1)^{s+2} n + (-1)^{t-1} d_{t-1} + \ldots d_1}_{(***)}\Bigg).
\end{aligned}$$

By (7), we can write $(*) = x - b_0 + \delta_1$ with $|\delta_1| < \varepsilon$. Similarly, by (8), we can write $(**) = x - 12y - d_0 + \delta_2$ with $|\delta_2| < \varepsilon$. And by design $(***) = b_0 - d_0$ so, putting these all together gives

$$s\left(\frac{h}{k}\right) = y + \frac{1}{12}(\delta_1 - \delta_2)$$
$$|y - s\left(\frac{h}{k}\right)| \le \varepsilon/2.$$

Also, by (7), we have $\left|x - \frac{h}{k}\right| < \varepsilon/2$ which gives (6) which concludes the proof. $\square$

While this was the first result showing the connection between continued fractions and Dedekind sums, since then more pleasing formulas have been found. While Hickerson in Theorem 6 focuses on the quotients in the Euclidean algorithm, here is one from Apostol which instead focuses on the remainders.

Theorem 7. [1, p. 72-73] Let, as above in (3), $0 < u_0 < u_1$ and $u_2, u_3, \dots, u_{j+1}$ be the series of remainders from the Euclidean algorithm for u_0 and u_1. Then

$$s(a, b) = \frac{1}{12}\sum_{j=1}^{n+1}\left((-1)^{j+1}\frac{u_j^2 + u_{j-1}^2 + 1}{u_j u_{j-1}}\right) - \frac{(-1)^n + 1}{8}.$$

This concludes our discussion of the fundamental arithmetic results about Dedekind sums. Using these results, we have a good understanding of how the Dedekind sum behaves and how to efficiently compute them. In addition, we have seen how they depend on the continued fraction, which encodes interesting information about the relationship between two arguments. We will see this again in the next section. Now, we look at exciting applications and connections to number theory and lattice point enumeration.

2 Connections to Other Fields

2.1 Connections to Number Theory

In 1877, Dedekind [1] defined the following function for $\tau \in H = \{z = x + iy \in \mathbb{C} : y > 0\}$

$$\eta(\tau) = e^{\pi i \tau/12}\prod_{n=1}^{\infty}(1 - e^{2\pi i n \tau}).$$

For τ in the upper half plane, $|e^{2\pi i n \tau}| < 1$ so the infinite product converges uniformly to non-zero values. Since this convergence is uniform, $\eta(\tau)$ is an analytic function where it's defined. Now, let

$$\Gamma = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1, a, b, c, d \in \mathbb{Z}\right\}/\{\pm 1\}$$

be the special linear group $\mathrm{PSL}_2(\mathbb{Z})$, sometimes called the modular group [18, p. 591].

Now, we briefly show that as usual, the composition of the modular transformation works well under matrix multiplication. Specifically, if $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$ then the map $\tau \to \frac{a_{11}\tau+a_{12}}{a_{21}\tau+a_{22}}$ is the same as the composition of the map $\tau \to \frac{b_{11}\tau+b_{12}}{b_{21}\tau+b_{22}}$ composed with $\tau \to \frac{c_{11}\tau+c_{12}}{c_{21}\tau+c_{22}}$. Indeed

$$\begin{aligned}\frac{b_{11}\left(\frac{c_{11}\tau+c_{12}}{c_{21}\tau+c_{22}}\right)+b_{12}}{b_{21}\left(\frac{c_{11}\tau+c_{12}}{c_{21}\tau+c_{22}}\right)+b_{22}} &= \frac{b_{11}c_{11}\tau+b_{11}c_{12}+b_{12}c_{21}\tau+b_{12}c_{22}}{b_{21}c_{11}\tau+b_{21}c_{12}+b_{22}c_{21}\tau+b_{22}c_{22}} \\ &= \frac{(b_{11}c_{11}+b_{12}c_{21})\tau+(b_{11}c_{12}+b_{12}c_{22})}{(b_{21}c_{11}+b_{22}c_{21})\tau+(b_{21}c_{12}+b_{22}c_{22})} = \frac{a_{11}\tau+a_{12}}{a_{21}\tau+a_{22}}.\end{aligned}$$

Dedekind studied the logarithm of $\eta(\tau)$. Since $\eta(\tau)$ is always non-zero, we take the principle branch of the logarithm of η and we obtain

$$\log\eta(\tau) = \frac{\pi i\tau}{12} - \sum_{m=1}^{\infty}\sum_{r=1}^{\infty}\frac{1}{r}e^{2\pi i\tau mr}. \tag{9}$$

Dedekind then proved the following:

Theorem 8. For $\begin{bmatrix} a & b \\ c & d\end{bmatrix} \in \Gamma$, and $c > 0$ we have

$$\log\eta\left(\frac{a\tau+b}{c\tau+d}\right) = \log\eta(\tau) + \frac{1}{2}\log\left(\frac{c\tau+d}{i}\right) + \pi i\frac{a+d}{12c} - \pi i s(d,c)$$

where the function $s(d,c)$ is the Dedekind sum.

For a proof, see [8]. In fact, this transformation formula is the origin of the Dedekind sum. By substituting different elements of Γ into this equation, Dedekind used this expression to prove Theorem 1 as follows.

Proof of Theorem 1. (From [16].) By simple matrix multiplication, we can see that

$$\begin{bmatrix} a & b \\ c & d\end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0\end{bmatrix} = \begin{bmatrix} b & -b \\ d & -c\end{bmatrix}. \tag{10}$$

We use this to obtain two expressions for $\log\eta\left(\frac{b\tau-a}{d\tau-c}\right)$. First we do this directly using (8):

$$\log\eta\left(\frac{b\tau-a}{d\tau-c}\right) = \log\eta(\tau) + \frac{1}{2}\log\left(\frac{d\tau-c}{i}\right) + \pi i\frac{b-c}{12d} - \pi i s(-c,d). \tag{11}$$

Now we compute

$$\begin{aligned}\log\eta(-1/\tau) = \log\eta\left(\frac{0-1}{\tau+0}\right) &= \log\eta(\tau) + \frac{1}{2}\log\left(\frac{\tau}{i}\right) + \pi i\frac{0}{12c} - \pi i s(0,1) \\ &= \log\eta(\tau) + \frac{1}{2}\log\left(\frac{\tau}{i}\right).\end{aligned}$$

Then, using (10), we have

$$\begin{aligned}\log\eta\left(\frac{b\tau-a}{d\tau-c}\right) &= \log\eta\left(\frac{a(-1/\tau)+b}{c(-1/\tau)+d}\right)\\ &= \log\eta(-1/\tau)+\frac{1}{2}\log\left(\frac{c(-1/\tau)+d}{i}\right)+\pi i\frac{a+d}{12c}-\pi i s(d,c)\\ &= \log\eta(\tau)+\frac{1}{2}\log\left(\frac{\tau}{i}\right)+\frac{1}{2}\log\left(\frac{-c/\tau+d}{i}\right)+\pi i\frac{a+d}{12c}-\pi i s(d,c)\\ &= \log\eta(\tau)+\frac{1}{2}\log\left(\frac{\tau}{i}\right)+\frac{1}{2}\log\left(\frac{\tau(-c/\tau+d)}{i^2}\right)+\pi i\frac{a+d}{12c}-\pi i s(d,c)\\ &= \log\eta(\tau)+\frac{1}{2}\log\left(\frac{\tau}{i}\right)+\frac{1}{2}\log\left(\frac{-c+d\tau)}{i^2}\right)+\pi i\frac{a+d}{12c}-\pi i s(d,c)\\ &= \log\eta(\tau)+\frac{1}{2}\log\left(\frac{\tau}{i}\right)+\frac{1}{2}\log\left(c-d\tau)\right)+\pi i\frac{a+d}{12c}-\pi i s(d,c) \qquad (12)\end{aligned}$$

Now we compare (11) and (12) to get

$$\begin{aligned}\pi i(s(d,c)-s(-c,d)) &= \frac{\pi i}{12}\left(\frac{a+d}{c}-\frac{b-c}{d}\right)+\frac{1}{2}\log\left(\frac{c-d\tau}{d\tau+c}i\right)\\ &= \frac{\pi i}{12}\left(\frac{ad+d^2-bc+c^2}{cd}\right)+\frac{1}{2}\log(-i)\\ &= \frac{\pi i}{12}\left(\frac{d}{c}+\frac{c}{d}+\frac{ad-bc}{cd}\right)+\frac{1}{2}\log(-i)\\ &= \frac{\pi i}{12}\left(\frac{d}{c}+\frac{c}{d}+\frac{1}{cd}\right)+\frac{1}{2}\log(-i).\end{aligned}$$

On this branch of the logarithm, $\log(-i)=-\frac{\pi}{4}$ and by Proposition 1 (1), $s(-c,d)=s(c,d)$ so we get $s(d,c)+s(c,d)=\frac{1}{12}\left(\frac{d}{c}+\frac{c}{d}+\frac{1}{cd}\right)-\frac{1}{4}$ as desired. □

This proof does not immediately seem to shed much insight on why Dedekind sums obey this reciprocity law. However, we see here another connection between Dedekind sums and the Euclidean algorithm. We know that

$$\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} \text{ and } \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$$

generate $\mathrm{PSL}_2(\mathbb{Z})$ [18, p. 594]. In addition, it easy to see that $\log\eta(\tau+1)=\log\eta(\tau)+\frac{\pi i}{12}$. With more work, one can show that $\log\eta(-1/\tau)=\log\eta(\tau)+\frac{1}{2}\log\left(\frac{\tau}{i}\right)$ (this is used in the proof of (8)). As is outlined in [19], expressing elements of Γ in terms of these generators is intimately connected to the Euclidean algorithm and continued fraction decompositions. The fact that the Dedekind Sum appears as the left-over piece in (8) demonstrates this connection.

Now we shift our attention to another area of number theory where Dedekind sums appear, namely quadratic reciprocity and the Jacobi symbol. First, we define, for p prime and a is an integer,

$$\left(\frac{a}{p}\right)\equiv a^{\frac{p-1}{2}} \pmod{p}.$$

Equivalently,

$$(a/p) = \begin{cases} 1 & a \text{ is a qudratic residue} \\ -1 & a \text{ is a qudratic non-residue} \\ 0 & p|a. \end{cases}$$

Extending this to odd composite $b = p_1 \dots p_k$, where the p_i are prime and $(a, b) = 1$, we define the Jacobi symbol to be

$$\left(\frac{a}{b}\right) = \prod_i \left(\frac{a}{p_i}\right).$$

Then, we have the following theorem first proved by Dedekind.

Theorem 9. [16] For b odd and $(a, b) = 1$,

$$12bs(a, b) \equiv b + 1 - 2\left(\frac{a}{b}\right) \pmod 8.$$

We will need the following lemma, which hints at the connection between the Legendre (and Jacobi) symbol and the sequence $a, 2a, \dots, (b-1)a$ that appears in the Dedekind sum formula.

Lemma 2. Let $m = \sum_{k=1}^{\frac{b-1}{2}} \left\lfloor \frac{2ka}{b} \right\rfloor$. Then

$$\left(\frac{a}{b}\right) = (-1)^m.$$

Although we don't give a full proof of this here, here is a brief outline from [16]. It relies on Gauss' criterion which is as follows: if b is an odd prime and $(a, b) = 1$, then $(a/b) = (-1)^m$ where m is the number of least positive remainders larger than $b/2$ in the sequence

$$a, 2a, \dots, \frac{b-1}{2}a \pmod b.$$

This can be generalized for the case where b is not prime, but merely positive and odd (see [3]). Then, one just needs to show that $\lfloor \frac{2ka}{b} \rfloor$ is odd if and only if ka is greater than $b/2$ when reduced mod b. A full proof of Lemma 2 can be found in [17].

Proof of Theorem 9. (from [16]) We start by directly computing

$$\begin{aligned} s(a, b) &= \sum_{k=0}^{b-1} \left(\left(\frac{k}{b}\right)\right)\left(\left(\frac{ka}{b}\right)\right) \\ &= \sum_{k=1}^{b-1} \frac{k}{b}\left(\frac{ka}{b} - \left\lfloor \frac{ka}{b} \right\rfloor - \frac{1}{2}\right) - \frac{1}{2}\underbrace{\sum_{k=1}^{b-1}\left(\left(\frac{ka}{b}\right)\right)}_{=0 \text{ by Lemma 1}} \end{aligned}$$

$$12bs(a, b) = 2a(b-1)(2b-1) - 12\sum_{k=1}^{b-1} k\left\lfloor \frac{ak}{b} \right\rfloor - 3b(b-1)$$

We now look at this expression mod 8. As b is odd, $b^2 \equiv 1 \mod 8$ so this becomes

$$\begin{aligned} 12bs(a,b) &\equiv 2a(3-3b) - 3(1-b) - 4\underbrace{\sum_{k=1}^{b-1} k \left\lfloor \frac{ak}{b} \right\rfloor}_{:=T} \\ &\equiv (6a-3)(1-b) - 4T \pmod 8. \end{aligned} \tag{13}$$

To determine this mod 8, we need to find T mod 2. Therefore, we can disregard the terms where k is even as those do not change the parity of the sum. So, for the odd terms, write $k = 2v - 1$ and we have

$$\begin{aligned} T &\equiv \sum_{v=1}^{\frac{b-1}{2}} (2v-1) \left\lfloor \frac{a(2v-1)}{b} \right\rfloor \\ &\equiv \sum_{v=1}^{\frac{b-1}{2}} \left\lfloor \frac{a(2v-1)}{b} \right\rfloor \pmod 2. \end{aligned}$$

This sum of the odd k terms can be obtained by taking all the terms and subtracting off the even terms to get

$$T \equiv \sum_{k=1}^{b-1} \left\lfloor \frac{ak}{b} \right\rfloor - \sum_{k=1}^{\frac{b-1}{2}} \left\lfloor \frac{2ka}{b} \right\rfloor \pmod 2.$$

The second term is m from Lemma 2 which means that $m = \frac{1}{2}((\frac{a}{b}) - 1) \mod 2$. The first term of this is not hard to handle. Indeed, since $(a,b) = 1$, as k runs through the non-zero residues mod b, so does ak so we get

$$\sum_{k=1}^{b-1} \left\lfloor \frac{ak}{b} \right\rfloor = \sum_{k=1}^{b-1} \frac{ak}{b} - \left\{ \frac{ak}{b} \right\} = \frac{a(b-1)}{2} - \sum_{k=1}^{b-1} \frac{k}{b} = \frac{a(b-1)}{2} - \frac{b-1}{2} = \frac{(b-1)(a-1)}{2}$$

where $\{x\} = x - \lfloor x \rfloor$.

Putting these facts into (13) gives

$$\begin{aligned} 12bs(a,b) &\equiv (6a-3)(1-b) - 2\left((a-1)(b-1) + \left(\frac{a}{b}\right) - 1\right) \\ &= (6a-3)(1-b) + 2(a-1)(1-b) - 2\left(\frac{a}{b}\right) + 2 \\ &= (8a-5)(1-b) - 2\left(\frac{a}{b}\right) + 2 \\ &\equiv -5 + 5b - 2\left(\frac{a}{b}\right) + 2 \\ &= 4(b-1) + b + 1 + 2\left(\frac{a}{b}\right) \\ &\equiv b + 1 + 2\left(\frac{a}{b}\right) \pmod 8. && \text{(since } b-1 \text{ is even)} \end{aligned}$$

□

This nice connection between the Jacobi symbol and Dedekind sums allows us to see a connection between Dedekind reciprocity and quadratic reciprocity. In particular, using Dedekind reciprocity, quadratic reciprocity is an easy corollary as follows.

Corollary 1 (Quadratic Reciprocity for the Jacobi symbol)**.** For odd coprime a, b

$$\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = (-1)^{\frac{a-1}{2}\frac{b-1}{2}}$$

Proof. Dedekind reciprocity states that for $(a,b) = 1$, we have $s(a,b) + s(b,a) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{b}{a} + \frac{1}{ab}\right)$. Multiplying by $12b$ we get

$$\begin{aligned} -3ab + a^2 + b^2 + 1 &= 12ab(s(a,b) + s(b,a)) \\ &\equiv ab + a - 2a\left(\frac{a}{b}\right) + ba + b - 2b\left(\frac{b}{a}\right) && \text{(Theorem 9)} \\ 2\left[a\left(\frac{a}{b}\right) + b\left(\frac{b}{a}\right)\right] &\equiv 5ab + 5 + a + b \pmod 8 && (14) \end{aligned}$$

since $a^2 \equiv b^2 \equiv 1 \pmod 8$. If $a \equiv b \equiv 3 \pmod 4$, then mod 4, (14) becomes

$$2\left[-\left(\frac{a}{b}\right) - b\left(\frac{b}{a}\right)\right] \equiv 1 + 1 - 1 - 1 = 0 \pmod 4$$

so $(\frac{a}{b}) + (\frac{b}{a}) \equiv 0 \pmod 2$ which means that $(\frac{a}{b})(\frac{b}{a}) = -1$ since $(\frac{a}{b}), (\frac{b}{a}) \in \{1, -1\}$. And, in this case $(-1)^{\frac{a-1}{2}\frac{b-1}{2}} = -1$ so it works.
On the other hand, now suppose $a \equiv 1$ or $b \equiv 1 \pmod 4$. We assume without loss of generality that $b \equiv 1 \pmod 4$ so we write $b = 4m + 1$. Then, (14) becomes

$$\begin{aligned} 2a\left(\frac{a}{b}\right) + 2\left(\frac{b}{a}\right) &\equiv 5a(4m+1) + 5 + a + 4m + 1 \\ &\equiv 4am + 4m - 2a - 2 \pmod 8 \\ &= 2(2m-1)(a+1) \end{aligned}$$

We reduce to looking at this mod 4:

$$\begin{aligned} a\left(\frac{a}{b}\right) + \left(\frac{b}{a}\right) &\equiv (2m-1)(a+1) \\ &\equiv -(a+1) && \text{(since } a \text{ is odd, } 2(a+1) \equiv 0 \pmod 4) \\ a\left[\left(\frac{a}{b}\right) + 1\right] + 1 + \left(\frac{b}{a}\right) &\equiv 0 \pmod 4. && (15) \end{aligned}$$

If $a \equiv -1 \pmod 4$, then (15) becomes $\left(\frac{a}{b}\right) \equiv \left(\frac{b}{a}\right) \pmod 4$.
If $a \equiv 1 \pmod 4$, then $\left(\frac{a}{b}\right) + \left(\frac{b}{a}\right) \equiv \pm 2 \pmod 4$. In either case, since $\left(\frac{a}{b}\right)$ and $\left(\frac{b}{a}\right)$ only take on the values -1 or 1, $\left(\frac{a}{b}\right) = \left(\frac{b}{a}\right)$ so $\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = 1$. And, in this case $(-1)^{\frac{a-1}{2}\frac{b-1}{2}} = 1$ so it works. $\square$

Theorems such as Theorem 9 which seek to understand Dedekind sum values modulo a certain number have appeared and continue to appear frequently in the literature. In fact, Apostol explains in his book on Modular Functions and Dirichlet Series that understanding the arithmetical nature of Dedekind sums is often important for understanding modular functions. In particular, the following theorem, which we don't prove, plays an important role in studying the invariance of modular functions under certain modular transformations [1].

Theorem 10. Let $q = 3, 5, 7$, or 13 and let $r = \frac{24}{q-1}$. Given integers a, b, c, d with $ad - bc = 1$ such that $c = c_1 q$ with $c_1 > 0$ the following is an even integer

$$s(a, c) - \frac{a+d}{12c} - s(a, c_1) - \frac{a+d}{c_1}.$$

We finish this section by noting that in 2018, Kohnen [11] published a proof using Dirichlet's theorem on primes in arithmetic progressions along with quadratic reciprocity to give another proof of Theorem 4 about the density of Dedekind sums on the real line. This proof is much shorter than the one given by Hickerson [7] and doesn't use continued fractions. Here we see that powerful number theory tools are being used to prove things about Dedekind sums, demonstrating again the tight connection between Dedekind sums and number theory.

2.2 Connections to Integer Lattice Point Enumeration in Polytopes

We will now present a connection between Dedekind sums and counting lattice points inside of rational polytopes. This is a rich subject that has been explored deeply (see for example [4]).

We start with a classical connection from [16]. Namely, we first give another proof of Theorem 1, by counting integer points in a certain three-dimensional parallelepiped. As noted in [16], there are proofs of quadratic reciprocity involving counting integer points in the plane, so this is a nice higher dimensional analogue, showing how Dedekind sums are in some sense a generalization of the Jacobi symbol.

Proof of Theorem 1. (from [16])

This proof is not short and requires some complex but unenlightening computations. So, we omit some. The full proof can be fonud in [16]. In particular, we assume the following equation:

$$\sum_{r=1}^{ab-1} \left\lfloor \frac{r}{a} \right\rfloor \left\lfloor \frac{r}{b} \right\rfloor = \frac{1}{12}(a-1)(b-1)(4ab+a+b+1) \tag{16}$$

In addition, by applying Lemma 1 and doing some manipulation we can see that Theorem 1 is equivalent to

$$12a \sum_{k=1}^{b-1} k \left\lfloor \frac{ak}{b} \right\rfloor + 12b \sum_{v=1}^{a-1} v \left\lfloor \frac{bv}{a} \right\rfloor = (a-1)(b-1)(8ab-a-b-1) \tag{17}$$

so we prove this.

We now proceed by counting the integer points in the orthogonal parallelepiped $ABCDEFGH$ (see Figure 1) with side length ab between A and E, side length a between A and D, and side length b between A and B. Divide the parallelepiped into three pyramids with planes AFG, ACG, and AGH.

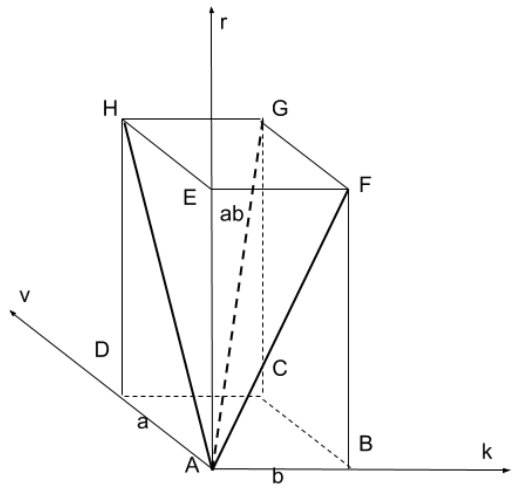


Figure 1: Parallelepiped $ABCDEFGH$ [16]

We will count the number of integer lattice points inside of the parallelepiped in two ways. First, we know that it is $(h-1)(k-1)(hk-1)$.
Now, we use our planes and pyramids to count this number again. Since $(a,b)=1$, the interior of the line between A and C contains no lattice points. Therefore, the plane ACG which lies entirely above this line, must contain no lattice points in its interior.
However the planes AFG and AGH do. Projecting the lattice points down onto the plane $ABCD$, we see that there are $(a-1)(b-1)$. This is because there are clearly $(a-1)(b-1)$ lattice points in the plane $ABCD$ and this projection is a bijection. In other words, if we have a lattice point in AFG or AGH, then it's projection onto $ABCD$ will give a lattice point there. Conversely, every lattice point in $ABCD$, when projected up onto AFG and AGH, gives a lattice point since the planes have an integer slope.
Now consider the two pyramids $A(BFGC)$ and $A(DCGH)$. We can count the lattice points in these pyramids by looking at planes parallel to their bases.
In $A(BFGC)$ the rectangle in the plane at distance $k \in \mathbb{Z}$ from A has base ak/b and height ka. Therefore, it has $\left\lfloor \frac{ak}{b} \right\rfloor ka$ lattice points on its interior. Summing these together, we see that the interior of $A(BFGC)$ has $a\sum_{k=1}^{b-1} b \left\lfloor \frac{ak}{b} \right\rfloor$ lattice points when counting the boundaries of the pyramid on the interior of the parallelepiped but not those on the boundary of the parallelepiped. Similarly, there are $k\sum_{v=1}^{a-1} v \left\lfloor \frac{bv}{a} \right\rfloor$ lattice points in the pyramid $A(DCGC)$ again counting only those lattice points on the interior of the parallelepiped.
Finally, we must count the number of lattice points in the pyramid $A(EFGH)$. To do this, we consider the planes parallel to $EFGH$ and see that the number of lattice points is $\sum_{r=1}^{hk-1} \left\lfloor \frac{r}{a} \right\rfloor \left\lfloor \frac{r}{b} \right\rfloor$ since for each $r \in \mathbb{Z}$, the number of lattice points in the square is $\left\lfloor \frac{r}{a} \right\rfloor \left\lfloor \frac{r}{b} \right\rfloor$, again counting only those lattice points on the interior of the parallelepiped.

Now, except for counting the $(a-1)(b-1)$ points in the planes AGH and AFG twice, we have counted all of the lattice points on the interior of the parallelepiped. Putting this together gives

$$b\sum_{v=1}^{a-1} v\left\lfloor\frac{bv}{a}\right\rfloor + a\sum_{k=1}^{b-1} k\left\lfloor\frac{ak}{b}\right\rfloor + \sum_{r=1}^{hk-1}\left\lfloor\frac{r}{a}\right\rfloor\left\lfloor\frac{r}{b}\right\rfloor - (a-1)(b-1) = (a-1)(b-1)(ab-1)$$

Using (16), multiplying by 12, and rearranging shows (17) and concludes the proof:

$$\begin{aligned} 12a\sum_{k=1}^{b-1} k\left\lfloor\frac{ak}{b}\right\rfloor + 12b\sum_{v=1}^{a-1} v\left\lfloor\frac{bv}{a}\right\rfloor &= 12ab(a-1)(b-1) - (a-1)(b-1)(4ab+a+b+1) \\ &= (a-1)(b-1)(8ab-a-b-1). \end{aligned}$$

□

We shift from using integer point enumeration to prove things about Dedekind sums to realizing that Dedekind sums and their generalization, Fourier-Dedekind sums, play a pivotal role in enabling us to systematically count integer points in polytopes.

As we have seen, there are many similarities and connections between the Dedekind sum and the greatest common divisor function. Here, we see that in many ways the Dedekind sum is a higher dimensional analogue to the GCD.

We start with the simple problem of counting integer points in integer vertex triangles. We restrict our attention to finding the number of integer lattice points in the following set for positive integers a and b:

$$\{(x,y)\in\mathbb{Z}^2_{\geq 0} : \frac{x}{a}+\frac{y}{b}\leq 1\}.$$

This is simply the triangle (see Figure 2) with vertices at the origin, $(a,0)$ and $(0,b)$.

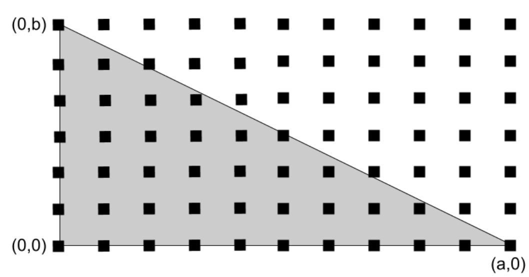


Figure 2: Lattice Points inside a Triangle

The number of lattice points in this triangle is given by

$$\frac{(a+1)(b+1)+1+\gcd(a,b)}{2}$$

since there are $\gcd(a,b)+1$ lattice points on the line connecting $(a,0)$ to $(0,b)$ and $(a+1)(b+1)$ is the total number lattice points in the rectangle of side length a and b.

In addition, as stated in [4, p. 6], if we want to know how many integers points lie on the one dimensional polytope $[0, a]$ for any real number a, the answer is clearly $\lfloor a \rfloor + 1$.

The following theorem from [4, p. 46] generalizes both of these examples and shows how the resulting Fourier-Dedekind sum builds upon both the ideas of the greatest common divisor function and the floor function. It will turn out that the Fourier-Dedekind sums that appear are natural generalizations of our Dedekind sums.

Theorem 11. Let $a, b, d, e, f, r \in \mathbb{Z}$ with $(e, f) = 1$ and $ea + fb \le rd$. Consider the following triangle (see Figure 3)

$$\mathcal{T} = \left\{ (x, y) \in \mathbb{R}^2 : x \ge \frac{a}{d}, y \ge \frac{b}{d}, ex + fy \le r \right\}$$

and define the *lattice point enumerator* for $\mathcal{T}$, which counts the lattice points in the t^{th} dilate of $\mathcal{T}$:

$$L_{\mathcal{T}}(t) = \left| \left\{ (m, n) \in \mathbb{Z}^2 : m \ge \frac{ta}{d}, n \ge \frac{tb}{d}, em + fn \le tr \right\} \right|.$$

Then,

$$\begin{aligned} L_{\mathcal{T}}(t) = {} & \frac{1}{2ef}(tr - u - v)^2 + \frac{1}{2}(tr - u - v)\left(\frac{1}{e} + \frac{1}{f} + \frac{1}{ef}\right) + \frac{1}{4}\left(1 + \frac{1}{e} + \frac{1}{f}\right) \\ & + \frac{1}{12}\left(\frac{e}{f} + \frac{f}{e} + \frac{1}{ef}\right) + \frac{1}{e}\sum_{j=1}^{e-1} \frac{\xi_e^{j(v-tr)}}{\left(1 - \xi_e^{jf}\right)\left(1 - \xi_e^{j}\right)} + \frac{1}{f}\sum_{l=1}^{f-1} \frac{\xi_f^{l(u-tr)}}{\left(1 - \xi_f^{le}\right)\left(1 - \xi_f^{l}\right)}. \end{aligned} \quad (18)$$

where $u = \left\lceil \frac{ta}{d} \right\rceil e$ and $v = \left\lceil \frac{tb}{d} \right\rceil f$.

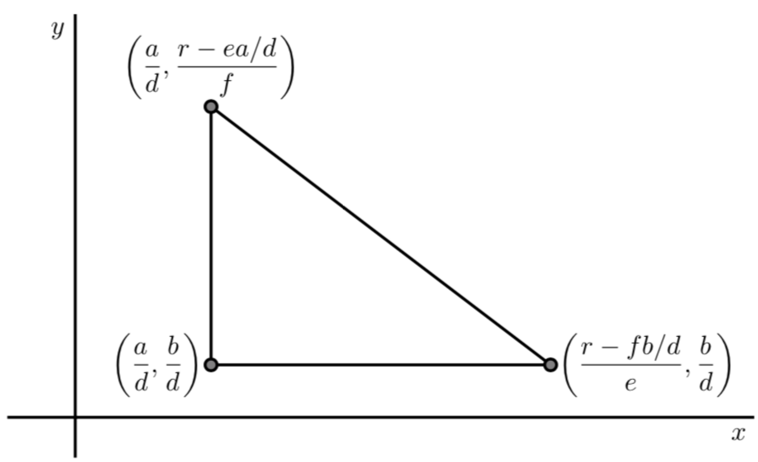


Figure 3: Rational Triangle, $\mathcal{T}$, from Theorem 11 [4, p. 44]

Proof. We start by re-writing the inequality in $L_{\mathcal{T}}(t)$ as an equality by introducing a slack variable:

$$L_{\mathcal{T}}(t) = \left|\left\{(m,n,s) \in \mathbb{Z}^3 : m \geq \frac{ta}{d}, n \geq \frac{tb}{d}, em + fn + s = tr\right\}\right|.$$

This number is the z^{tr} coefficient in the following function

$$\left(\sum_{m \geq \lceil \frac{ta}{d} \rceil} z^{em}\right)\left(\sum_{n \geq \lceil \frac{tb}{d} \rceil} z^{fn}\right)\left(\sum_{s \geq 0} z^s\right).$$

These sums are over the integers which allows us to replace $\frac{ta}{d}$ with $\lceil \frac{ta}{d} \rceil$ and $\frac{tb}{d}$ with $\lceil \frac{tb}{d} \rceil$ in the summation bound.

Now, using geometric series, $\sum_{i \geq n} z^i = \frac{z^n}{1-z}$ for $|z| < 1$ and writing, for example, z^{em} as $(z^e)^m$, this becomes

$$\frac{(z^e)^{\lceil \frac{ta}{d} \rceil}}{1-z^e}\frac{(z^f)^{\lceil \frac{tb}{d} \rceil}}{1-z^f}\frac{1}{1-z} = \frac{z^{u+v}}{(1-z^e)(1-z^f)(1-z)}. \tag{19}$$

Note that we can check that $u + v - tr - e - f - 1 < 0$ so the expression in (19) is indeed a proper rational function which means we can apply the machinery of partial fractions.

To find the z^{tr} term of (19), we have two options. We could find the residue at $z = 0$ of (19) divided by z^{tr+1} using the residue theorem on circles of radius $r > 1$ centered at the origin (here, we once again would need that $u + v - tre - f - 1 - 1 < -1$ in order to make the integral along the circle vanish). On the other hand, the method used primarily in [4] (and which we follow) is to use partial fractions to find the constant term of (19) divided by z^{tr} (taking $z \neq 0$) to see that

$$\begin{aligned} L_{\mathcal{T}}(t) &= \operatorname{const}\left(\frac{z^{u+v-tr}}{(1-z^e)(1-z^f)(1-z)}\right) = \operatorname{const}\left(\frac{1}{(1-z^e)(1-z^f)(1-z)z^{tr-u-v}}\right) \\ &= \operatorname{const}\left(\sum_{j=1}^{e-1}\frac{A_j}{z-\xi_e^j} + \sum_{l=1}^{f-1}\frac{B_l}{z-\xi_f^l} + \sum_{k=1}^{3}\frac{C_k}{(z-1)^k} + \sum_{k=1}^{tr-u-v}\frac{D_k}{z^k}\right). \end{aligned}$$

It turns out that the D_k terms do not contribute to the constant term because they contribute only to the terms in the Laurent series with negative exponents. Therefore we can find $L_{\mathcal{T}}(t)$ by evaluating the function without the D_k terms at $z = 0$ which gives

$$L_{\mathcal{T}}(t) = -\sum_{j=1}^{e-1}\frac{A_j}{\xi_e^j} - \sum_{l=1}^{f-1}\frac{B_l}{\xi_e^l} - C_1 + C_2 - C_3 \tag{20}$$

One can calculate these coefficients by some manipulation and taking the limit as z approaches the relevant pole. Then, the result is achieved by just plugging in the calculated values into (20). For the details, see [4, p. 46].

□

The last two terms in (18) motivate us to define the Fourier-Dedekind Sum. Let $\xi_b = e^{2\pi i/b}$ be a bth root of unity. Then define [4, p. 15]

$$s_n(a_1, \ldots, a_d; b) = \frac{1}{b} \sum_{k=1}^{b-1} \frac{\xi_b^{kn}}{\left(1 - \xi_b^{ka_1}\right)\left(1 - \xi_b^{ka_2}\right) \cdots \left(1 - \xi_b^{ka_d}\right)}.$$

These appear frequently in the study of the restricted partition function and the Frobeneous problem [4] in addition to their uses in counting lattice points in rational polytopes.

The goal now is to explain how Fourier-Dedekind sums are a natural generalization of the standard Dedekind sum. To do that, we show that Dedekind sums can be expressed by way of finite Fourier series. The finite Fourier series is a beautiful subject which can be used to express complex-valued periodic functions on the integers as polynomials in the b^{th} roots of unity $\xi^n = e^{2\pi i n/b}$. For more on finite Fourier series, [4, Chapter 7] is very instructive.

As a way of introducing and motivating the Fourier series, we give an example from [4, p. 136] for the Fourier series for a simple periodic function.

Proposition 4. Let $a(n)$ be periodic with period 3 given by $a(0) = 1, a(1) = 5, a(2) = 2$. Then, if $\xi = e^{2\pi i/3}$,

$$a(n) = \frac{8}{3} + \left(-\frac{4}{3} - \xi\right)\xi^n + \left(-\frac{1}{3} + \xi\right)\xi^{2n}.$$

This can be proved by defining the generating function $F(z) = \sum_{n\geq 0} a(n) z^n$ and expanding it into its partial fraction decomposition.

Now, we find something similar for Dedekind sums.

Proposition 5. For a, b relatively prime, let $\xi = e^{2\pi i/b}$. Then

$$s(a,b) = \frac{1}{4b} \sum_{k=1}^{b-1} \frac{1+\xi^k}{1-\xi^k} \frac{1+\xi^{-ka}}{1-\xi^{-ka}}.$$

Proof. (from [4, p. 139])

First we show that

$$\left(\left(\frac{a}{b}\right)\right) = \frac{1}{2b} \sum_{k=1}^{b-1} \frac{1+\xi^k}{1-\xi^k} \xi^{ak}. \tag{21}$$

To show this, we rely on the machinery of the finite Fourier transform. That is, we can express $\left(\left(\frac{a}{b}\right)\right) = \sum_{k=0}^{b-1} \hat{a}(k) \xi^{ak}$ where

$$\hat{a}(k) = \frac{1}{b} \sum_{m=0}^{b-1} \left(\left(\frac{m}{b}\right)\right) \xi^{-mk}.$$

By Lemma 1, $\hat{a}(0) = 0$. For $k \neq 0$,

$$\hat{a}(k) = \frac{1}{b} \sum_{m=1}^{b-1} \left(\frac{m}{b} - \frac{1}{2}\right) \xi^{-mk} = \frac{1}{b^2} \sum_{m=1}^{b-1} m \xi^{-mk} + \frac{1}{2b} = \frac{1}{b}\left(\frac{\xi^k}{1-\xi^k} + \frac{1}{2}\right) = \frac{1}{2b} \frac{1+\xi^k}{1-\xi^k}.$$

Here we are using the fact [4, p. 146] that for $(a,b)=1$, $\frac{1}{b}\sum_{k=1}^{b-1} k\xi^{-ak} = \frac{\xi^a}{1-\xi^a}$. This establishes (21).

Now, we compute

$$\begin{aligned}
s(a,b) = \sum_{k=0}^{b-1}\left(\left(\frac{ka}{b}\right)\right)\left(\left(\frac{k}{b}\right)\right) &= \frac{1}{(2b)^2}\sum_{k=0}^{b-1}\left[\left(\sum_{\mu=1}^{b-1}\frac{1+\xi^\mu}{1-\xi^\mu}\xi^{\mu ka}\right)\left(\sum_{\nu=1}^{b-1}\frac{1+\xi^\nu}{1-\xi^\nu}\xi^{\nu ka}\right)\right] \\
&= \frac{1}{4b^2}\sum_{k=0}^{b-1}\left[\sum_{\mu=1}^{b-1}\sum_{\nu=1}^{b-1}\frac{1+\xi^\mu}{1-\xi^\mu}\frac{1+\xi^\nu}{1-\xi^\nu}\xi^{\mu ka+vk}\right] \\
&= \frac{1}{4b^2}\sum_{\mu=1}^{b-1}\sum_{\nu=1}^{b-1}\frac{1+\xi^\mu}{1-\xi^\mu}\frac{1+\xi^\nu}{1-\xi^\nu}\sum_{k=0}^{b-1}\xi^{\mu ka+vk}.
\end{aligned}$$

The last sum over k is equal to 0 for $\nu+\mu a \not\equiv 0 \pmod{b}$ since $\xi^{k(\nu+\mu a)} = (\xi^{\nu+\mu a})^k$. So, the only time we get a non-zero term is when $v \equiv -\mu a \pmod{b}$. In that case, $\xi^{k(\nu+\mu a)} = 1$ so the sum becomes b. Thus, we can write

$$s(a,b) = \frac{1}{4b^2}\sum_{\mu=1}^{b-1}\frac{1+\xi^\mu}{1-\xi^\mu}\frac{1+\xi^{-\mu a}}{1-\xi^{-\mu a}}\cdot b = \frac{1}{4b}\sum_{k=1}^{b-1}\frac{1+\xi^k}{1-\xi^k}\frac{1+\xi^{-ka}}{1-\xi^{-ka}}.$$

□

Given this, we are ready to show how the Fourier-Dedekind sum is a generalization of the Dedekind sum with the following proposition.

Proposition 6. For a,b relatively prime integers, $s_0(a,1;b) = -s(a,b) + \frac{b-1}{4}$.

The proof of this is available in [4, p. 150] and is just a case of manipulating sums.

We continue this study by investigating a three-dimensional, integer vertex, analogue to the earlier problem about counting lattice points in the triangle using the greatest common divisor which is due to L. J. Mordell [13] through the following Theorem.

Theorem 12. [13] Let a,b,c be be pairwise coprime positive integers, and let $N_3(a,b,c)$ be the number of lattice points in the tetrahedron, $\mathcal{P}$, defined by the following planes:

$$0 \le x, y, z\,;\; \frac{x}{a}+\frac{y}{b}+\frac{z}{c} \le 1.$$

Then,

$$\begin{aligned}
N_3(a,b,c) = -(s(bc,a), +s(ca,b), +s(ab,c)) + \frac{abc}{6} + \frac{ab+ca+bc+a+b+c+4}{4} \\
+\frac{1}{12}\left(\frac{bc}{a}+\frac{ac}{b}+\frac{ab}{c}+\frac{1}{abc}\right)+1.
\end{aligned}$$

Note that here $\mathcal{P}$ is the tetrahedron with vertices at $(a,0,0), (0,b,0)$, and $(0,0,c)$.

This theorem was originally proved in [13] with a tricky counting argument. Instead of showing that method, we briefly outline a more recent proof from [4, p. 158] that extends the result. This uses very similar methods as used in our proof of Theorem 11.

Proof outline. We compute the lattice point enumerator for this tetrahedron:

$$\begin{aligned} L_{\mathcal{P}}(t) &= \left| \left\{ (k,l,m) \in \mathbb{Z}^3 : k,l,m \geq 0, \frac{k}{a} + \frac{l}{b} + \frac{m}{c} \leq t \right\} \right| \\ &= \left| \left\{ (k,l,m,n) \in \mathbb{Z}^4 : k,l,m,n \geq 0, bck + acl + abm + n = abct \right\} \right| \end{aligned}$$

with the insertion of a slack variable to turn this three-dimensional polytope in three-space into a three-dimensional hyperplane in four dimensional space.

We again notice that this is exactly the z^{abct} term of

$$\left(\sum_{k \geq 0} z^{bck} \right) \left(\sum_{l \geq 0} z^{acl} \right) \left(\sum_{m \geq 0} z^{abm} \right) \left(\sum_{n \geq 0} z^{n} \right) = \frac{1}{(1-z^{bc})(1-z^{ac})(1-z^{ab})(1-z)}. \tag{22}$$

and use similar machinery of partial fractions to show that

$$\begin{aligned} L_{\mathcal{P}}(t) = \frac{abc}{6}t^3 + \frac{ab+ac+bcc+1}{4}t^2 + \left(\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} + \frac{1}{abc} \right) t \\ + (s_0(bc,1;a) + s_0(ca,1;b) + s_0(ab,1;c))t + 1. \end{aligned} \tag{23}$$

Using Proposition 6, Theorem 12 follows easily. □

In fact (23) is actually a much more general result which shows an example of how the lattice point enumerator function for a polytope is a polynomial when the polytope has integer vertices. This, along with more, is the content of Ehrhart's theorem, a fascinating result which connects a polytope's lattice point enumerator to its volume and reveals many more deep and interesting connections. For more on this, see [4, Chapter 3].

We finish this section by stating a result that generalizes these ideas to higher dimensions.

Counting lattice points in polytopes can be naturally applied to computing the following *restricted partition function*

$$p_A(n) = \left| \left\{ (m_1, \ldots, m_d) \in \mathbb{Z}^d : m_j \geq 0, m_1 a_1 + \cdots + m_d a_d = n \right\} \right|.$$

It turns out, that by using similar methods to in Theorem 11, we can compute this as a Fourier-Dedekind sum with the following theorem

Theorem 13. (from [4, p. 15])

$$\begin{aligned} p_A(n) = -B_1 + B_2 - \cdots + (-1)^d B_d + s_{-n}(a_2, a_3, \ldots, a_d; a_1) + s_{-n}(a_1, a_3, a_4, \ldots, a_d; a_2) \\ + \cdots + s_{-n}(a_1, \ldots, a_{d-1}; a_d) \end{aligned}$$

where the B_i are computable constants.

This concludes our discussion the role Dedekind sums play in number theory and in counting lattice points inside polytopes. We have seen a strong connection between Dedekind sums and the Jacobi symbol, and seen how important understanding Dedekind sums in analyzing transformations of the Dedekind η function, a key function in analytic number theory. In addition, we have seen how Fourier-Dedekind sums, a natural generalization of Dedekind sums appears in enumerating lattice points. This led us to realize how Dedekind sums can be represented as sums over the b^{th} roots of unity using the finite Fourier transform.

3 Recent Questions

In this last section, we discuss modern approaches to two open problems.

3.1 What Values do Dedekind Sums Take?

We are now concerned with finding the exact set $\{s(a,b) : a, b \in \mathbb{Z}, b > 0\}$, an open question according to [4, p. 165].

Girstmair [6] proved the following necessary condition of which fractions $\frac{k}{q}$ appear as values of normalized Dedekind sums.

Theorem 14. [6] Suppose for integers a, b, k, q with $b \geq 1, q \geq 2$ and $(a, b) = (k, q) = 1$, it holds that $12s(a, b) = \frac{k}{q}$. Then,

- If $3 \not| q$, then $3|k$.
- If $2 \not| q$, then

$$k \equiv \begin{cases} 2 \pmod 4 & q \equiv 3 \pmod 4 \\ 0 \pmod 8 & q \text{ is a square} \\ 0 \pmod 4 & \text{otherwise.} \end{cases} \tag{24}$$

And, he conjectured the following converse.

Conjecture 1. [6] For integers k, q with $q \geq 2, (k, q) = 1$, there exists integers a, b such that $b \geq 1$, $(a, b) = 1$ such that

$$12s(a,b) = \frac{k}{q}$$

if and only if the conditions in Theorem 14 hold.

It would be nice if we could prove this conjecture, as that would tell us what the set $\{s(a, b) : a, b \in \mathbb{Z}\}$ is. While the full proof of this does not seem currently available, Michael Kural recently [12] proved the conjecture for certain values of q.

Theorem 15. [12] Conjecture 1 holds for q even or a square divisible by 3 or 5.

This is nice because it allows us to once again prove Theorem 4, this time quite easily.

Proof of Theorem 4. Since the rationals are dense in $\mathbb{R}$, let $\frac{m}{n}$ with $(m, n) = 1$ be any rational number. Given $\varepsilon > 0$ we need to find $(a, b) = 1$ such that $|12s(a, b) - \frac{m}{n}| \leq \varepsilon$. By Theorem 15, let $q \in 7$ such that $6|q$. Then q is even and the conditions in 14 hold vacuously true. So, we may find a, b such that $12s(a, b) = k/q$ for any such q and all k. So, let $q > \frac{1}{\varepsilon}$ such that $6n^2|q$. And, let $k = \frac{mq}{n} + 1$. Since $n^2|q$, we know that $\frac{mq}{n}$ is an integer divisible by q, so $(k, q) = 1$. Now, it suffices to check

$$\begin{aligned} \left|\frac{k}{q} - \frac{m}{n}\right| &= \left|\frac{\frac{mq}{n} + 1}{q} - \frac{m}{n}\right| \\ &= \left|\frac{m}{n} - \frac{m}{n} + \frac{1}{q}\right| = \frac{1}{q} < \varepsilon. \end{aligned}$$

$\square$

3.2 What pairs a, a' exist such that $s(a,b) = s(a',b)$?

Another open question, according to Beck and Robins [4, p. 164], is to find, for a given b, all pairs of integers $a, a' \in \mathbb{Z}_b$ such that $s(a,b) = s(a',b)$. We know that by Proposition 1, $s(a,b) = s(a',b)$ if $aa' \equiv 1 \pmod{b}$ or $a \equiv a' \pmod{b}$. And, for primes, we have the following converse mentioned earlier.

Proposition 7. [4, p. 162] For p prime, $s(a,p) = s(a',p)$ if and only if if $aa' \equiv 1 \pmod{p}$ or $a \equiv a' \pmod{p}$.

However, by simple computation, we see that $s(4,25) = s(9,25)$ and yet $4 \cdot 9 \equiv 11 \pmod{25}$. Motivated by this and many other examples, we make the following conjecture for the case where $b = p^2$ for p prime.

Conjecture 2. Let p be prime. If $a, a' \in \mathbb{Z}$ such that $(a, p^2) = (a', p^2) = 1$, then $s(a', p^2) = s(a', p^2)$ if and only if at least one of the following holds

- $a \equiv a' \pmod{p^2}$
- $aa' \equiv 1 \pmod{p^2}$
- $a \equiv a' \equiv 1 \pmod{p}$ and $a, a' \not\equiv 1 \pmod{p^2}$
- $a \equiv a' \equiv -1 \pmod{p}$ and $a, a' \not\equiv -1 \pmod{p^2}$.

While a full proof of this is currently unavailable, we can easily verify it to be true up to $p \leq 13$ and it appears to hold for larger primes also.

This concludes our discussion of recent efforts to tackle two open problems. It seems as if Conjecture 1 very well could be true, which would answer our first question. And, it seems likely that Conjecture 2 could be true. However, the case for general b in our second question seems erratic and hard to understand. For example, even going to the case where $b = p^3$ exhibits surprising irregularity.

In this paper, we have explored the behavior of Dedekind sums through elementary properties and continued fractions. We have seen two of the major areas from which study of Dedekind sums both benefits and informs. Finally, we have seen progress towards two questions that still remain unanswered.

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